Conformal Invariance and Duality in Self-Dual Gravity and (2,1) Heterotic String Theory

M. Abou Zeid and C. M. Hull

Physics Department, Queen Mary and Westfield College,
Mile End Road, London E1 4NS, U. K.

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Abstract

A system of gravity coupled to a 2-form gauge field, a dilaton and Yang-Mills fields in $2n$ dimensions arises from the (2,1) sigma model or string. The field equations imply that the curvature with torsion and Yang-Mills field strength are self-dual in four dimensions, or satisfy generalised self-duality equations in $2n$ dimensions. The Born-Infeld-type action describing this system is simplified using an auxiliary metric and shown to be classically Weyl invariant only in four dimensions. A dual form of the action is found (no isometries are required). In four dimensions, the dual geometry is self-dual gravity without torsion coupled to a scalar field. In $D > 4$ dimensions, the dual geometry is hermitian and determined by a $D - 4$ form potential K, generalising the Kähler potential of the four dimensional case, with the fundamental 2-form given by $\tilde{J} = i*\partial\bar{\partial}K$. The coupling to Yang-Mills is through a term $K \wedge tr(F \wedge F)$ and leads to a Uhlenbeck-Yau field equation $\tilde{J}^{ij}F_{ij} = 0$.

1   Introduction

The superstring with (2,1) world-sheet supersymmetry provides important insights into M-theory and superstring theory. The target space of the (2,1) string is 2+2 dimensional, with a null reduction restricting the dynamics to 1+1 or 2+1 dimensions [1]. The target space dynamics has been shown in [2, 3, 4] to describe critical string worldsheets or membrane worldvolumes in static gauge and constitutes an explicit realisation of the scenario proposed in [5]. Furthermore, it was found that all types of ten-dimensional superstring theories and the eleven-dimensional super-membrane arise as vacua of the (2,1) heterotic string. More recently, Martinec [7, 8] has argued that the (2,1) string may provide the degrees of freedom needed to define certain compactifications of the matrix model of M-theory proposed in ref. [9].

The (2,1) heterotic string was shown in [1] to describe a theory of gravity with torsion coupled to Yang-Mills gauge fields in 2+2 dimensions. The null reduction mentioned above must be imposed, and yields a 1+1 dimensional space or a 2+1 dimensional space depending on the orientation of the null Killing vector used in the null reduction [1]. The field equations were found in [13, 14] and the effective action for the gravitational and Yang-Mills degrees of freedom (before null reduction) was obtained in refs. [6, 12]. The geometry is a generalisation of Kähler geometry with torsion [10] and a hypersymplectic structure [12], and the field equations imply that the curvature with torsion is self-dual in 2+2 dimensions. The Yang-Mills fields are also self-dual in 2+2 dimensions. In higher dimensions, the field equations imply that the curvature with torsion has $SU(n_1, n_2)$ holonomy, while the Yang-Mills fields satisfy a non-linear form of the Uhlenbeck-Yau equation [12]. The action in 10+2 dimensions is the effective space-time theory that is conjectured to give supergravity in 10+1 or 9+1 dimensions upon null reduction [6].

Our purpose here is twofold. In section 3, we will formulate an equivalent form of the (2,1) string action with an auxiliary metric and will show that it is Weyl invariant only in four dimensions. In section 4, we will dualise the vector potential that governs the geometry to find an equivalent action given in terms of a $D - 4$ form potential in D dimensions. In four dimensions, the dual geometry is Kähler.

2   (2,1) Geometry

We begin by recalling the geometric conditions for (2,1) supersymmetry of the (1,1) sigma model with metric g_{ij} and anti-symmetric tensor b_{ij} [10, 11] (see also refs. [14, 13]; further discussion of the geometry, isometry symmetries and gauging of the (2,1) model can be found in refs. [15, 16, 17]). The sigma model is invariant under (2,1) supersymmetry [10, 13, 14] if the target space is even dimensional ($D = 2n$) with a complex structure $J^i{}_j$ which is covariantly constant with respect to the connection with torsion $\Gamma^{(+)}$ and with respect to which the metric is hermitian, so that $J_{ij} \equiv g_{ik} J^k{}_j$ is antisymmetric.

It is useful to introduce complex coordinates $z^\alpha, \overline{z}^{\overline{\beta}}$ in which the line element is $ds^2 = 2g_{\alpha\overline{\beta}} dz^\alpha d\overline{z}^{\overline{\beta}}$ and the exterior derivative decomposes as $d = \partial + \overline{\partial}$. The con-

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ditions for (2,1) supersymmetry imply that H is given in terms of the fundamental 2-form

$$J = \frac{1}{2}J_{ij}d\phi^i \wedge d\phi^j = -ig_{\alpha\overline{\beta}}dz^\alpha \wedge d\overline{z}^{\overline{\beta}} \tag{1}$$

by

$$H = i(\partial - \overline{\partial})J. \tag{2}$$

Then the condition $dH = 0$ implies

$$i\partial\overline{\partial}J = 0 \tag{3}$$

so that locally there is a (1,0) form potential $k = k_\alpha dz^\alpha$ such that

$$J = i(\partial\overline{k} + \overline{\partial}k). \tag{4}$$

In a suitable gauge, the metric and torsion potential are then given by

$$g_{ij} = \begin{pmatrix} 0 & g_{\alpha\overline{\beta}} \\ g_{\overline{\alpha}\beta} & 0 \end{pmatrix} \quad , \quad b_{ij} = \begin{pmatrix} 0 & b_{\alpha\overline{\beta}} \\ b_{\overline{\alpha}\beta} & 0 \end{pmatrix} \tag{5}$$

so that the torsion (2) is given by

$$H_{\alpha\beta\overline{\gamma}} = \frac{1}{2}\left(g_{\alpha\overline{\gamma},\beta} - g_{\beta\overline{\gamma},\alpha}\right) \quad , \quad H_{\alpha\beta\gamma} = 0 \tag{6}$$

while the constraint (3) implies that the metric satisfies

$$g_{\alpha[\overline{\beta},\overline{\gamma}]\delta} - g_{\delta[\overline{\beta},\overline{\gamma}]\alpha} = 0. \tag{7}$$

From (4), the geometry is defined locally by

$$\begin{aligned} g_{\alpha\overline{\beta}} &= \partial_\alpha \overline{k}_{\overline{\beta}} + \overline{\partial}_{\overline{\beta}}k_\alpha \\ b_{\alpha\overline{\beta}} &= \partial_\alpha \overline{k}_{\overline{\beta}} - \overline{\partial}_{\overline{\beta}}k_\alpha. \end{aligned} \tag{8}$$

If $k_\alpha = \partial_\alpha K$ for some K, then the torsion vanishes and the manifold is Kähler with Kähler potential K, but if $dk \neq 0$ then the space is a hermitian manifold of the type introduced in [10].

The (1,1) supersymmetric model will be conformally invariant at one-loop if there is a function Φ such that

$$R_{ij}^{(+)} - \nabla_{(i}\nabla_{j)}\Phi - H^k{}_{ij}\nabla_k\Phi = 0 \tag{9}$$

where $R_{ij}^{(+)}$ is the Ricci tensor for the connection with torsion. The curvature and Ricci tensors with torsion are

$$R_{lij}^{(+)k} = \partial_i\Gamma_{jl}^{(+)k} - \partial_j\Gamma_{il}^{(+)k} + \Gamma_{im}^{(+)k}\Gamma_{jl}^{(+)m} - \Gamma_{jm}^{(+)k}\Gamma_{il}^{(+)m}, \quad R_{ij}^{(+)} = R_{ikj}^{(+)k}. \tag{10}$$

It will be useful to define the vector

$$v^i = H_{jkl}J^{ij}J^{kl} \tag{11}$$

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together with the $U(1)$ part of the curvature

$$C_{ij}^{(+)} = J^l{}_k R^{(+)k}{}_{lij} \tag{12}$$

and the $U(1)$ part of the connection

$$\Gamma_i^{(+)} = J^k{}_j \Gamma_{ik}^{(+)j} = i(\Gamma_{i\alpha}^{(+)\alpha} - \Gamma_{i\bar{\alpha}}^{+\bar{\alpha}}). \tag{13}$$

In a complex coordinate system, (12) can be written as

$$C_{ij}^{(+)} = \partial_i \Gamma_j^{(+)} - \partial_j \Gamma_i^{(+)}. \tag{14}$$

If the metric has Euclidean signature, then the holonomy of any metric connection (including $\Gamma^{(+)}$) is contained in $O(2n)$, while if it has signature $(2n_1, 2n_2)$ where $n_1 + n_2 = n$, it will be in $O(2n_1, 2n_2)$. The holonomy $\mathcal{H}(\Gamma^{(+)})$ of the connection $\Gamma^{(+)}$ is contained in $U(n_1, n_2)$. It will be contained in $SU(n_1, n_2)$ if in addition

$$C_{ij}^{(+)} = 0 \tag{15}$$

where the $U(1)$ part of the curvature is given by (12). As C_{ij} is a representative of the first Chern class, a necessary condition for this is the vanishing of the first Chern class.

It was shown in [13] that geometries for which

$$\Gamma_i^{(+)} = 0 \tag{16}$$

in some suitable choice of coordinate system will satisfy the one-loop conditions (9) provided the dilaton is chosen as

$$\Phi = -\frac{1}{2} \log |\det g_{\alpha\bar{\beta}}|, \tag{17}$$

which implies

$$\partial_i \Phi = v_i. \tag{18}$$

Moreover, the one-loop dilaton field equation is then satisfied for compact manifolds, or for non-compact ones in which $\nabla\Phi$ falls off sufficiently fast [12]. This implies that $\mathcal{H}(\Gamma^{(+)}) \subseteq SU(n_1, n_2)$ and these geometries generalise the Kähler Ricci-flat or Calabi-Yau geometries, and reduce to these in the special case in which $H = 0$. These are not the most general solutions of (9) [12].

The condition that the connection $\Gamma^{(+)}$ has $SU(n_1, n_2)$ holonomy can be cast as a generalised self-duality condition on the curvature $R^{(+)}$. Defining the four-form

$$\phi^{ijkl} \equiv -3J^{[ij}J^{kl]} \tag{19}$$

the condition that $\mathcal{H}(\Gamma^{(+)}) \subseteq SU(n_1, n_2)$ is equivalent to

$$R_{ijkl}^{(+)} = \frac{1}{2} g_{im} g_{jn} \phi^{mnpq} R_{pqkl}^{(+)}. \tag{20}$$

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For $D = 4$, $\phi^{ijkl} = -\epsilon^{ijkl}$ and this is the usual anti-self-duality condition, while for $D > 4$, this is an example of the generalised self-duality equations considered for Riemannian manifolds in refs. [18, 19, 20, 21].

The equation (16) can be viewed as a field equation for the potential k_α. It can be obtained by varying the action [6, 12]

$$S = \int d^D x \sqrt{|\det g_{\alpha\bar{\beta}}|} \tag{21}$$

where $g_{\alpha\bar{\beta}}$ is given in terms of k_α by (8). It follows from the form (5) of the metric that this action can be rewritten as

$$S = \int d^D x |\det g_{ij}|^{1/4} \tag{22}$$

which is non-covariant but is invariant under volume-preserving diffeomorphisms.

This can be generalised to include Yang-Mills fields A_i taking values in some group G, in addition to g_{ij} and b_{ij}. The A_i must be a connection for a holomorphic vector bundle (so that the field-strength F is a (1,1) form), with Chern-Simons form $\Omega(A)$ satisfying $d\Omega = tr F^2$, Bott-Chern form Υ [22] defined by

$$tr(F^2) = i\partial\bar{\partial}\Upsilon \tag{23}$$

and a form χ defined by

$$\Omega(A) = i(\partial - \bar{\partial})\Upsilon + d\chi. \tag{24}$$

The conditions for (2,1) supersymmetry and conformal invariance then imply the existence of a (1,0) form k such that (4) is replaced by

$$J = \Upsilon + i(\partial\bar{k} + \bar{\partial}k) \tag{25}$$

and the metric and torsion potential are given by

$$
\begin{aligned}
g_{\alpha\bar{\beta}} &= i\Upsilon_{\alpha\bar{\beta}} + \partial_\alpha \bar{k}_{\bar{\beta}} + \bar{\partial}_{\bar{\beta}} k_\alpha \\
b_{\alpha\bar{\beta}} &= i\chi_{\alpha\bar{\beta}} + \partial_\alpha k_{\bar{\beta}} - \bar{\partial}_{\bar{\beta}} k_\alpha.
\end{aligned}
\tag{26}
$$

The field equations can again be obtained by varying the action (21), but with $g_{\alpha\bar{\beta}}$ given by (26). The Yang-Mills equation is

$$J^{ij} F_{ij} = 0. \tag{27}$$

It is sometimes useful to write the metric in terms of a fixed background metric $\hat{g}_{\alpha\beta}$ (e. g. a flat metric) which is given in terms of a potential $\hat{k}$ by $\hat{g}_{\alpha\bar{\beta}} = \bar{\partial}_{\bar{\beta}}\hat{k}_\alpha + \partial_\alpha \hat{\bar{k}}_{\bar{\beta}} + i\Upsilon_{\alpha\bar{\beta}}$, and a fluctuation given in terms of a vector field B_i defined by

$$B_\alpha = i(k_\alpha - \hat{k}_\alpha) \quad , \quad B_{\bar{\alpha}} = -i(\bar{k}_{\bar{\alpha}} - \hat{\bar{k}}_{\bar{\alpha}}) \tag{28}$$

with field strength $\mathcal{F}_{ij} = 2\partial_{[i}B_{j]}$. Then

$$g_{\alpha\bar{\beta}} = \hat{g}_{\alpha\bar{\beta}} + i\mathcal{F}_{\alpha\bar{\beta}}. \tag{29}$$

The action (21) becomes

$$S = \int d^D x \sqrt{|\det(\hat{g}_{\alpha\bar{\beta}} + i\mathcal{F}_{\alpha\bar{\beta}})|} \tag{30}$$

which is similar to a Born-Infeld action and is invariant under the abelian gauge symmetry $\delta B = d\lambda$.

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3 Weyl-Invariant Action for (2,1) Strings

The Nambu-Goto action for the bosonic string can be rewritten using an auxiliary world-sheet metric [23, 24] in a way which is useful for many purposes, such as quantization [25]. Similarly, the action (22) can be written in the classically equivalent alternative form

$$S' = T'_4 \int d^D x |\gamma|^{1/4} \left[\gamma^{ij} g_{ij} - (D-4)c \right] \tag{31}$$

where γ_{ij} is an auxiliary metric, $\gamma = \det \gamma_{ij}$ and c, T' are (real) constants. The field equation for γ_{ij} is

$$\gamma_{ij} = \frac{1}{c} g_{ij} \tag{32}$$

for $D \neq 4$, and

$$\gamma_{ij} = \frac{4}{(\gamma^{kl} g_{kl})} g_{ij} \tag{33}$$

for $D = 4$. Substituting back in (31) one recovers the action (22) with the constant T' given by

$$T'_4 = \frac{1}{4} c^{\frac{D}{4}-1}. \tag{34}$$

In complex coordinates, the action (31) takes the form

$$S' = T' \int d^D x \sqrt{|\gamma|} \left[\gamma^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}} + \gamma^{\bar{\alpha}\beta} g_{\bar{\alpha}\beta} - (D-4)c \right] \tag{35}$$

where now $\gamma = \det \gamma_{\alpha\bar{\beta}}$. The field equations (32) or (33) imply that the components $\gamma^{\alpha\beta}$ and $\gamma^{\bar{\alpha}\bar{\beta}}$ vanish, so that on-shell the auxiliary metric γ_{ij} is hermitian,

$$J^j{}_{(i} \gamma_{k)j} = 0. \tag{36}$$

It is then consistent to impose the condition (36) that the metric be hermitian off-shell as well, and we shall do so in what follows.

The action (35) is a special case of the general class of action

$$S' = T'_q \int d^D x |\gamma|^{1/q} \left[\gamma^{ij} g_{ij} - (D-q)c \right]. \tag{37}$$

For $D \neq q$, the field equation for the auxiliary tensor is (32), and substituting this back in (37) yields actions of the form

$$S = \int d^D x |\det(g_{ij})|^{1/q} \tag{38}$$

with the constant T' given by

$$T'_q = \frac{1}{q} c^{\frac{D}{q}-1}. \tag{39}$$

In the special case in which $D = q$, the constant term in the action (37) vanishes and there is a generalised Weyl symmetry under

$$\gamma_{ij} \to \omega(x) \gamma_{ij}. \tag{40}$$

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The field equation in this case is

$$\gamma_{ij} = \frac{q}{(\gamma^{kl}g_{kl})}g_{ij}. \tag{41}$$

For all D, q there is in addition an invariance under volume preserving diffeomorphisms, i.e. diffeomorphisms of the D-dimensional space-time which preserve $\det \gamma_{ij}$, so that the vector field ξ^i generating the diffeomorphism must satisfy $\nabla_i \xi^i = 0$ where $\nabla_i \xi^i = \gamma^{-1/2}\partial_i(\gamma^{1/2}\xi^i)$ and ∇_i is the usual covariant derivative for the metric γ_{ij}. For $D = q$, the symmetry consists of the volume preserving diffeomorphisms, together with the Weyl transformations.

4   Duality

We now discuss the dualisation of the vector potential k_i, starting with the simplest case of four space-time dimensions and no background metric or Yang-Mills fields. The discussion will be generalised below to include the background metric and the coupling to the Yang-Mills fields.

Consider the action (21) and add a Lagrange multiplier term imposing the constraint (8),

$$S = \int d^4x \left[\sqrt{|g|} - \frac{1}{4}\left\{ \Lambda^{\alpha\overline{\beta}}\left(g_{\alpha\overline{\beta}} - \partial_\alpha \overline{k}_{\overline{\beta}} - \overline{\partial}_{\overline{\beta}} k_\alpha\right) + c.c. \right\} \right] \tag{42}$$

with $g \equiv \det g_{\alpha\overline{\beta}}$ (the sign of the Lagrange term is arbitrary). Eliminating $\Lambda^{\alpha\overline{\beta}}$ from (42), we recover the action (21) subject to the constraint (8). Alternatively, we can first integrate over the vectors $k_\alpha, \overline{k}_{\overline{\alpha}}$, which are Lagrange multipliers for the constraints

$$\begin{aligned}
\partial_\alpha \Lambda^{\alpha\overline{\beta}} &= 0 \\
\overline{\partial}_{\overline{\beta}} \Lambda^{\alpha\overline{\beta}} &= 0.
\end{aligned} \tag{43}$$

In $D = 4$ dimensions, these can be solved locally in terms of a scalar K :

$$\Lambda^{\alpha\overline{\beta}} = L^{\alpha\overline{\beta}} \tag{44}$$

where $L^{\alpha\overline{\beta}}$ is the 'field strength' of K given by

$$L^{\alpha\overline{\beta}} \equiv \epsilon^{\alpha\gamma\overline{\beta}\overline{\delta}}\partial_\gamma \overline{\partial}_{\overline{\delta}} K \tag{45}$$

and $\epsilon^{\alpha\gamma\overline{\beta}\overline{\delta}}$ is the antisymmetric tensor density (with $\epsilon^{1\overline{1}2\overline{2}} = 1$). Then integrating over $k, \overline{k}$ and solving as in (44), the action takes the form

$$S = \int d^4x \left[\sqrt{|g|} - \frac{1}{4}\left\{ L^{\alpha\overline{\beta}}g_{\alpha\overline{\beta}} + c.c. \right\} \right]. \tag{46}$$

Note that in (42) we have chosen $\Lambda^{\alpha\overline{\beta}}$ to be a tensor density so that the second term in the action is fully diffeomorphism invariant, even though the first term is only

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invariant under volume preserving diffeomorphisms. This proves to be the most convenient choice, but equivalent results could have been obtained by choosing $\Lambda^{\alpha\overline{\beta}}$ to transform differently, so that the second term in (42) was also only invariant under volume preserving diffeomorphisms.

Alternatively, one can add a Lagrange multiplier term imposing the constraint (7),

$$S = \int d^4x \left[\sqrt{|g|} - \frac{1}{4} \left\{ \epsilon^{\alpha\gamma\overline{\beta}\overline{\delta}} K \partial_\gamma \partial_{\overline{\delta}} g_{\alpha\overline{\beta}} + c.c. \right\} \right] \tag{47}$$

Integrating by parts yields

$$S = \int d^4x \left[\sqrt{|g|} - \frac{1}{4} \left\{ \epsilon^{\alpha\gamma\overline{\beta}\overline{\delta}} \partial_\gamma \partial_{\overline{\delta}} K g_{\alpha\overline{\beta}} + c.c. \right\} \right] \tag{48}$$

which by definition (45) of $L^{\alpha\overline{\beta}}$ is identical to action (46). Thus it is equivalent to impose either of the constraints (8) or (7).

The field equation for $g_{\alpha\overline{\beta}}$ which follows from (46) is

$$\sqrt{|g|} g^{\alpha\overline{\beta}} = L^{\alpha\overline{\beta}}. \tag{49}$$

Taking determinants in (49) yields the constraint

$$\det L^{\alpha\overline{\beta}} = -1 \tag{50}$$

for signature (2,2) or $\det L^{\alpha\overline{\beta}} = 1$ for signature (4,0). Taking the trace gives

$$L^{\alpha\overline{\beta}} g_{\alpha\overline{\beta}} = 2\sqrt{|g|}. \tag{51}$$

Substituting (51) back into the action (46), a cancellation occurs and the action vanishes,

$$S = 0. \tag{52}$$

The dynamics is contained entirely in the constraint (50). Consider the Kähler metric $G_{\alpha\overline{\beta}}$ with potential K,

$$G_{\alpha\overline{\beta}} \equiv \partial_\alpha \overline{\partial}_{\overline{\beta}} K. \tag{53}$$

Then (50) implies

$$\det G_{\alpha\overline{\beta}} = -1 \tag{54}$$

for signature (2,2), or $\det G_{\alpha\overline{\beta}} = 1$ for signature (4,0). Thus the dual metric $G_{\alpha\overline{\beta}}$ is Kähler and Ricci-flat.

The equation (49) implies

$$g_{\alpha\overline{\beta}} = \Omega L_{\alpha\overline{\beta}} \tag{55}$$

for some scalar field Ω, and the constraint (7) will be satisfied if (54) and

$$G^{\alpha\overline{\beta}} \partial_\alpha \partial_{\overline{\beta}} \Omega = 0 \tag{56}$$

hold. Then Ω is a harmonic scalar on the dual space.

Writing $G_{\alpha\bar\beta} = \eta_{\alpha\bar\beta} + \partial_\alpha\bar\partial_{\bar\beta}\varphi$ where $\eta_{\alpha\bar\beta}$ is a flat background metric, (54) becomes the following equation for φ:

$$\det \begin{pmatrix} 1 + \partial_1\bar\partial_1\varphi & \partial_1\bar\partial_2\varphi \\ \partial_2\bar\partial_1\varphi & -1 + \partial_2\bar\partial_2\varphi \end{pmatrix} = 1 \tag{57}$$

in the notation of [1]. This equation and (56) can be derived from the action

$$\int \partial\varphi\bar\partial\varphi + \frac{1}{3!}\varphi\bar\partial\varphi \wedge \partial\bar\partial\varphi + \int \sqrt{G}G^{\alpha\bar\beta}\partial_\alpha\Omega\partial_{\bar\beta}\Omega, \tag{58}$$

where the first term is the Plebanski action (also with the notation of [1]) and the second term is such that (56) is the field equation obtained by varying Ω and using the constraint (54). The action (58) can be thought of as the dual action.

We have thus established the following result. We started with the theory of hermitian gravity with torsion in four dimensions defined by the action (21), the field equations of which implied that the curvature with torsion was anti-self-dual and with holonomy $SU(2)$ (for signature (4,0)) or $SL(2,R)$ (for signature (2,2)). We then dualised this to obtain anti-self-dual Riemannian gravity coupled to a harmonic scalar Ω, with no torsion and the action (58). Thus in four dimensions, a theory with torsion is related by a conformal rescaling to a theory without torsion. This is in agreement with the results of [33, 34]. We emphasize that this duality (unlike the dualities considered e.g. in [30, 11, 31, 32]) does not require any Killing vectors.

The generalisation to other (even) dimensions is straightforward. Consider the action

$$S = \int d^D x \left[\sqrt{|g|} - \frac{1}{4}\left\{ \Lambda^{\alpha\bar\beta}\left(g_{\alpha\bar\beta} - \partial_\alpha\bar k_{\bar\beta} - \bar\partial_{\bar\beta}k_\alpha \right) + c.c. \right\} \right]. \tag{59}$$

Eliminating $\Lambda^{\alpha\bar\beta}$ from (59), we recover the action (21). Alternatively, integrating out k_α, $\bar k_{\bar\alpha}$ gives the constraints (43). The solution to (43) in $D = 2n$ dimensions is

$$\Lambda^{\alpha\bar\beta} = L^{\alpha\bar\beta} \tag{60}$$

where

$$L^{\alpha\bar\beta} \equiv \epsilon^{\alpha\gamma_1\ldots\gamma_{n-1}\bar\beta\bar\delta_1\ldots\bar\delta_{n-1}}\partial_{\gamma_1}\bar\partial_{\bar\delta_1}K_{\gamma_2\ldots\gamma_{n-1}\bar\delta_2\ldots\bar\delta_{n-1}} \tag{61}$$

is the 'field strength' of an $(n-2, n-2)$ form K. The action then takes the form

$$S = \int d^D x \left[\sqrt{|g|} - \frac{1}{4}\left\{ L^{\alpha\bar\beta}g_{\alpha\bar\beta} + c.c. \right\} \right] \tag{62}$$

with L given by (61). The field equation for $g_{\alpha\bar\beta}$ is

$$\sqrt{|g|}g^{\alpha\bar\beta} = L^{\alpha\bar\beta}. \tag{63}$$

Taking determinants in (63), we find

$$\sqrt{|g|} = \left| \det L^{\alpha\bar\beta} \right|^{\frac{1}{n-2}} \tag{64}$$

Contracting (63) with $g_{\alpha\bar\beta}$ yields

$$L^{\alpha\bar\beta} g_{\alpha\bar\beta} = n \left| \det L^{\alpha\bar\beta} \right|^{\frac{1}{n-2}} . \tag{65}$$

It is easily checked that the solution of the field equation (63) is of the form

$$g^{\alpha\bar\beta} = \mu \left| \det L^{\alpha\bar\beta} \right|^{\nu} L^{\alpha\bar\beta} \tag{66}$$

where μ and ν are constants given by

$$\mu = 1 \quad , \quad \nu = -\frac{1}{n-2}. \tag{67}$$

Substituting (64) and (65) into (62) gives the dual action

$$S = -\frac{1}{2}(n-2) \int d^D x \left| \det L^{\alpha\bar\beta} \right|^{\frac{1}{n-2}} \tag{68}$$

for the field strength L of the $D-4$ form potential K. Again, we have chosen the Lagrange multiplier to be a tensor density so that the Lagrange multiplier term in the action is coordinate invariant.

We now reinstate the fixed background $\hat g_{\alpha\bar\beta}$, which will be taken to be of the form $\hat g_{\alpha\bar\beta} = \overline\partial_{\bar\beta} \hat k_\alpha + \partial_\alpha \hat{\bar k}_{\bar\beta} + i\Upsilon_{\alpha\bar\beta}$ and includes the coupling to Yang-Mills fields, through $\Upsilon_{\alpha\bar\beta}$. As a result of (23), this background metric satisfies

$$\hat g_{\alpha[\bar\beta,\bar\gamma]\delta} - \hat g_{\delta[\bar\beta,\bar\gamma]\alpha} = -4 F_{\alpha[\bar\beta} F_{\bar\gamma]\delta}. \tag{69}$$

Consider the action

$$S = \int d^D x \left[\sqrt{|g|} - \frac{1}{4} \left\{ \Lambda^{\alpha\bar\beta} \left(g_{\alpha\bar\beta} - \hat g_{\alpha\bar\beta} - \partial_\alpha \bar k_{\bar\beta} - \overline\partial_{\bar\beta} k_\alpha \right) + c.c. \right\} \right]. \tag{70}$$

Eliminating $\Lambda^{\alpha\bar\beta}$ from (70), we recover the action (21) (after shifting the potentials $k \to k - \hat k$). The vectors are Lagrange multipliers for the constraints (43), which can be solved locally in terms of a $D-4$ form K as in (60). On integrating out the vectors, the action takes the form

$$S = \int d^D x \left[\sqrt{|g|} - \frac{1}{4} \left\{ L^{\alpha\bar\beta} \left(g_{\alpha\bar\beta} - \hat g_{\alpha\bar\beta} \right) + c.c. \right\} \right] \tag{71}$$

where $L^{\alpha\bar\beta}$ is given in (61). Using the field equation for $g_{\alpha\bar\beta}$, taking the determinant and the trace and substituting back into the action (71), we find the dual action in the form

$$S = \frac{1}{4} \int d^D x \left[L^{\alpha\bar\beta} \hat g_{\alpha\bar\beta} + c.c. - 2(n-2) \left| \det L^{\alpha\bar\beta} \right|^{\frac{1}{n-2}} \right]. \tag{72}$$

$(n \neq 2)$. In the absence of Yang-Mills fields, $\Upsilon_{\alpha\bar\beta} = 0$, then the term $L^{ij} \hat g_{ij}$ in (72) vanishes after integration by parts, as a result of the form (61) of $L^{\alpha\bar\beta}$ and the fact that the background metric $\hat g_{\alpha\bar\beta}$ satisfies (7).

If the Yang-Mills fields do not vanish, then using the form (61) of $L^{\alpha\bar{\beta}}$, integrating by parts and using (69), we obtain

$$S = -\frac{1}{2}(n-2) \int d^D x \left|\det L^{\alpha\bar{\beta}}\right|^{\frac{1}{n-2}} - \frac{1}{2} \int K \wedge tr(F \wedge F) \tag{73}$$

with an interesting coupling of the $D-4$ form potential to $F \wedge F$.

For $n = 2$, integrating out the metric in (71) gives the constraint

$$\det L^{\alpha\bar{\beta}} = 1 \tag{74}$$

for signature (2,2), or $\det L^{\alpha\bar{\beta}} = -1$ for signature (4,0), while the action reduces to the term $\int K \wedge tr(F \wedge F)$. The constraint (74) can be imposed via a Lagrange multiplier Λ, so that the action becomes

$$S = -\frac{1}{2} \int K \wedge tr(F \wedge F) - \frac{1}{2} \int d^D x \Lambda \left(\det L^{\alpha\bar{\beta}} - 1\right). \tag{75}$$

Integrating out the scalar Λ yields the constraint (74), so that one recovers the action $\int K \wedge (F \wedge F)$ subject to this constraint. Instead we keep the Lagrange multiplier; using (23), (61) and integrating by parts, we find the following field equation for K

$$\partial\bar{\partial}\left(i\Upsilon - \Lambda \det(L^{\alpha\bar{\beta}})L^{-1}\right) = 0, \tag{76}$$

where L^{-1} is the 2-form $(L^{-1})_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^{\bar{\beta}}$. This implies that

$$\Lambda(\det L)L^{-1} = iJ' \tag{77}$$

where

$$J' = i(\partial\bar{k}' + \bar{\partial}k') + \Upsilon \tag{78}$$

for some (1,0) form potential k', so that J' is the 2-form corresponding to a metric $g'_{\alpha\bar{\beta}}$ defining some dual (2,1) sigma-model. Although (77) is not algebraic in K, one can solve for Λ as a functional of K, k' and F; taking determinants in (77), we find

$$\Lambda = \pm\sqrt{\frac{\det(g')}{\det L}}, \tag{79}$$

which can be substituted back in (75).

It is useful to define a dual metric $\tilde{g}_{\alpha\bar{\beta}}$ (for $n \neq 1$) by

$$\tilde{g}_{\alpha\bar{\beta}} \equiv \left|\det L^{\alpha\bar{\beta}}\right|^{\frac{1}{n-1}} \left(L^{-1}\right)_{\alpha\bar{\beta}}, \tag{80}$$

so that

$$L^{ij} = \sqrt{\det \tilde{g}_{ij}}\,\tilde{g}^{ij}. \tag{81}$$

Then the dual geometry is given in terms of the fundamental two-form

$$\tilde{J} = -i\tilde{g}_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^{\bar{\beta}} \tag{82}$$

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by

$$\tilde{J} = i * \partial \overline{\partial} K \tag{83}$$

where the Hodge star operation is defined with respect to the metric (80). The dual action (73) (for $n \neq 2$) can also be expressed in terms of the dual metric (80) and we find

$$S = -\frac{1}{2}(n-2) \int d^D x \left| \det \tilde{g}_{\alpha\bar{\beta}} \right|^{\frac{n-1}{n-2}} - \frac{1}{2} \int *\tilde{J} \wedge \Upsilon \tag{84}$$

using eqs. (23) and (83). The constraint (83) defines a class of hermitian geometries (without torsion) in which the metric is given in terms of a $D-4$ form potential instead of the scalar potential of Kähler geometry. The action (84) gives a field equation for such metrics which arises naturally from the dualisation of (2,1) geometry.

For $n = 2$, the Kähler form $J_G = i\partial\overline{\partial}K$ corresponding to the Kähler metric $G_{\alpha\bar{\beta}}$ defined in (53) is dual to $\tilde{J}$, $J_G = *\tilde{J}$, so that the action becomes

$$S = -\frac{1}{2} \int *\tilde{J} \wedge \Upsilon = -\frac{1}{2} \int J_G \wedge \Upsilon \tag{85}$$

subject to the constraint (83); this is the Donaldson action [27] for self-dual Yang-Mills in a self-dual geometry.

The field equation for the Yang-Mills fields derived from the action (84) is

$$\tilde{J}^{ij} F_{ij} = 0. \tag{86}$$

Thus F satisfies the Uhlenbeck-Yau equation with respect to the complex structure $\tilde{J}_{ij}$. This can be derived, for example, by transforming with a complex gauge transformation with parameter h taking values in the complexification G_c of the gauge group G [12],

$$F = h^{-1} f h \tag{87}$$

where

$$f = da + a^2 = \overline{\partial} a \tag{88}$$

is the field strength of a holomorphic connection given by the (1,0) form $a = U^{-1}\partial U$. Then using $tr F^2 = tr f^2$ and varying with respect to the prepotential U gives

$$\partial\overline{\partial}K \wedge f + \overline{\partial}K \wedge \partial f + \overline{\partial}K \wedge [a, f] = 0 \tag{89}$$

which gives (86) on using (83) and the Bianchi identity.

The field equations for the dual metric $\tilde{g}_{ij}$ and its implications for the dual geometry will be discussed elsewhere. Note that the dualisation procedure carried out in the foregoing can also be applied to the actions of section 3 with an auxiliary metric; the results are equivalent.

5 Conclusion

Summarizing, the (2,1) sigma-model or string give rise to a theory of gravity coupled to a two-form gauge field, a dilaton and Yang-Mills gauge fields in $D = 2n$ dimensions. The field equations imply that the curvature with torsion is self-dual in four dimensions, or has $SU(n)$ holonomy in $2n$ dimensions. The system is described by the Born-Infeld type action (21), where $g_{\alpha\bar{\beta}}$ is given in terms of k_α by (8). This action can be simplified using an auxiliary metric, and the forms (35) and (31) are classically equivalent to (30) and (22) respectively. The four-dimensional action (31) is classically invariant under diffeomorphisms preserving the volume element constructed from the auxiliary metric and under the generalised Weyl transformation (40). It would be interesting to compare this symmetry to the infinite dimensional current algebra [26] of the Donaldson action for self-dual Yang-Mills [27, 28].

The action (30) can be dualised, with no isometries being required. The dual theory in four dimensions is self-dual gravity *without* torsion coupled to a scalar. This recovers the remarkable equivalence between self-dual hermitian geometries with torsion and self-dual gravity without torsion. In higher dimensions, dualising gives an aparently new generalisation of Kähler geometry, in which the metric $\tilde{g}_{ij}$ is hermitian and is determined by the $(n-2, n-2)$ form potential K (which can be thought of as analogous to the scalar potential of Kähler geometry) via (61) and the dynamics is described by the action (84). The couplng to Yang-Mills is via the term $K \wedge F \wedge F$ and gives rise to the Uhlenbeck-Yau-type field equation (86).

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